WESLEY COLLEGE
By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER ONE 2019
TEST 3: Vectors

Name: Solutions

Monday 20th May

Time: 50 minutes

Total marks: $\frac{}{20} + \frac{}{30} = \frac{}{50}$

Calculator free section – maximum 15 minutes

1. [9 marks – 2, 3, 2, 1 and 1]

For the vectors $\vec{p} = 2i + j - 2k$ and $\vec{q} = 2i - 2j + k$

(a) show that $\vec{p}$ is perpendicular to $\vec{q}$

$\vec{p} \cdot \vec{q} = 4 - 2 - 2 = 0$ ✓

$\therefore \vec{p} \perp \vec{q}$ ✓

(b) calculate, in simplest form, a vector $\vec{r}$ that is perpendicular to both $\vec{p}$ and $\vec{q}$

$\vec{p} \times \vec{q} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ ✓ $\times \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ -6 \end{bmatrix}$ ✓ or $\vec{q} \times \vec{p}$

Simplest form $\vec{r} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ ✓

There is a plane that includes both $\vec{p}$ and $\vec{q}$ and passes through $A(1, -2, 3)$. Write an equation for this plane in:

(c) normal form

$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ ✓ $\cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = 3$ ✓ or equivalent

(d) Cartesian form

$x + 2y + 2z = 3$ ✓ or equivalent

(e) vector form $\vec{r} = \vec{a} + \lambda \vec{b} + \mu \vec{c}$

$\vec{r} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ ✓

2. [5 marks]

Calculate the point of intersection of the planes defined by the simultaneous system

$$\begin{cases} x+2y+3z=10 \\ 2x-y-z=3 \\ x+y+4z=9 \end{cases}$$

(1) − (3) $\quad y - z = 1$ ✓ — (4)

2(1) − (2) $\quad 5y + 7z = 17$ ✓

5(4) $\quad 5y - 5z = 5$

∴ $\quad 12z = 12$ ✓

$z = 1$

$y = 2$, $x = 3$ ✓

OR

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 4 & 9 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 0 & 3 & 9 & 15 \\ 1 & 1 & 4 & 9 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ 2R_3 - R_2 \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 12 & 12 \end{array}\right] R_2 - 3R_1$$

Point of intersection is $(3, 2, 1)$ ✓

3. [6 marks – 2, 1, 1 and 2]

When Gaussian elimination, using elementary row operations, was applied to a system of equations in variables x, y and z, this augmented echelon matrix resulted:

$$P = \left[\begin{array}{ccc|c} k^2-4 & 0 & 0 & k+2 \\ 1 & -2 & 0 & 3 \\ 1 & 3 & -2 & 8 \end{array}\right]$$

For which value(s) of k will:

(a) the system have no solutions ✓

$k^2 - 4 = 0$ and $k + 2 \neq 0$ $\Rightarrow k = 2$ ✓

(b) there be an infinite number of solutions

$k^2 - 4 = 0$ and $k + 2 = 0$ $\Rightarrow k = -2$ ✓

(c) $x = 1$? $\quad \dfrac{k+2}{k^2-4} = 1 \Rightarrow \dfrac{1}{k-2} = 1 \Rightarrow k = 3$ ✓

Hence:

(d) Evaluate y and z when $x = 1$

$x = 1 \Rightarrow y = -1$ ✓ $\left(\frac{3-1}{-2}\right)$

and $z = -5$ ✓ $\left(\frac{8-1+3}{-2}\right)$

Year 12 Specialist Test 3: Vectors

Name: ______________________

Time: 35 minutes 30 marks

Calculator assumed section

4. [6 marks – 2, 2, and 2]

Two points A and B have position vectors $\overrightarrow{OA} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$.

Determine:

(a) the exact area of ΔOAB

$$\text{Area} = \frac{1}{2}|\overrightarrow{OA}||\overrightarrow{OB}|\sin\theta$$

$$= \frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}| \checkmark$$

$$= \frac{3\sqrt{3}}{2} \checkmark \text{ (classpad)} \quad \text{or} \quad \frac{1}{2}\left|\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}\right|$$

(b) (measure or size of) $\angle AOB$

$$\text{Angle}(\overrightarrow{OA}, \checkmark \overrightarrow{OB}) = 158.2^\circ \checkmark \text{ (ClassPad)}$$

or solve $\overrightarrow{OA} \cdot \overrightarrow{OB} = -13 = \sqrt{14} \cdot \sqrt{14} \cos\theta$

(c) two different vectors of length $\sqrt{3}$ that are perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$

$$\left.\begin{aligned} \overrightarrow{OA} \times \overrightarrow{OB} &= \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \\ |\overrightarrow{OA} \times \overrightarrow{OB}| &= 3\sqrt{3} \end{aligned}\right\} \checkmark$$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \text{ and } -\frac{1}{3}\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \text{ are two such vectors}$$

$\checkmark$

5. [9 marks –1, 2, 1, 1, 2 and 2]

When Rodney, the human cannonball in the Kleenheat TV advertisement, is fired from his cannon, his position vector, at time t seconds $t \geq 0$, is defined, in metres, by $\overrightarrow{r(t)} = 10t\,i + (10 + 5t - 4.9t^2)\,j$

Determine:

(a) his velocity vector $\overrightarrow{v(t)}$

$\vec{v}(t) = 10i + (5 - 9.8t)j$ ✓

(b) his initial angle of elevation

$v(0) = 10i + 5j$ ✓

$\tan\theta = \frac{5}{10}$

$\theta = 26.6°$ ✓

~~(c) his velocity vector $\overrightarrow{v(t)}$~~

(c) his acceleration

$\vec{a}(t) = -9.8j$ ✓

(d) when he reaches the high point of his trajectory

When $5 - 9.8t = 0$

$t = 0.51$ seconds ✓

(e) his height (above ground level) when he hits his target 18 m (horizontally) from the cannon

$r(t) = 18i \Rightarrow t = 1.8$ ✓

height is j component : $h = 10 + 5 \times 1.8 - 4.9 \times 1.8^2$

$= 3.1$ m ✓ (3.124)

(f) the length of his trajectory

$= \int_0^{1.8} \left| 10i + (5 - 9.8t)j \right| dt = 21.09$ m ✓ ✓ (ClassPad)

6. [8 marks – 5, 1 and 2]

(a) Calculate the minimum distance from the point $A(-3,7,-8)$ to the plane $3x-4y+5z=23$

Normal to plane is $\begin{bmatrix}3\\-4\\5\end{bmatrix}$; perp line is $\vec{r}=\begin{bmatrix}-3\\7\\-8\end{bmatrix}+\lambda\begin{bmatrix}3\\-4\\5\end{bmatrix}$ ✓

Line intersects plane when $\begin{bmatrix}3\\-4\\5\end{bmatrix}\cdot\begin{bmatrix}3\lambda-3\\-4\lambda+7\\5\lambda-8\end{bmatrix}=23$ ✓

$\Rightarrow \quad 9\lambda-9+16\lambda-28+25\lambda-40=23$

$50\lambda-77=23$

$\lambda=2$ ✓

Point of contact $\begin{bmatrix}3\\-1\\2\end{bmatrix}$ ✓

and distance $\sqrt{6^2+8^2+10^2}=\sqrt{200}$ or $10\sqrt{2}$ units ✓

(or scalar projection or parallel plane or ....)

(b) Determine the equation of the sphere, centred at $A(-3,7,-8)$ that is tangential to $3x-4y+5z=23$

$(x+3)^2+(y-7)^2+(z+8)^2=200$ ✓

or $\left|\vec{r}-\begin{bmatrix}-3\\7\\-8\end{bmatrix}\right|=10\sqrt{2}$

(c) Identify another plane that is tangential to the sphere in (b) with the same minimum distance from $A(-3,7,-8)$.

Parallel plane through opposite end of diameter ✓

$3x-4y+5z=c$ at $(-9,15,-18)$ NA

$=-27-60-90$

$=-177$ ✓

or $\vec{r}\cdot\begin{bmatrix}3\\-4\\5\end{bmatrix}=-177$

7. [7 marks –2, 2, 1, 1 and 1]

An ice-skater follows a cyclic path with her velocity given by $\overrightarrow{v(t)} = 6\cos 2t\, i - 4\sin t\, j$

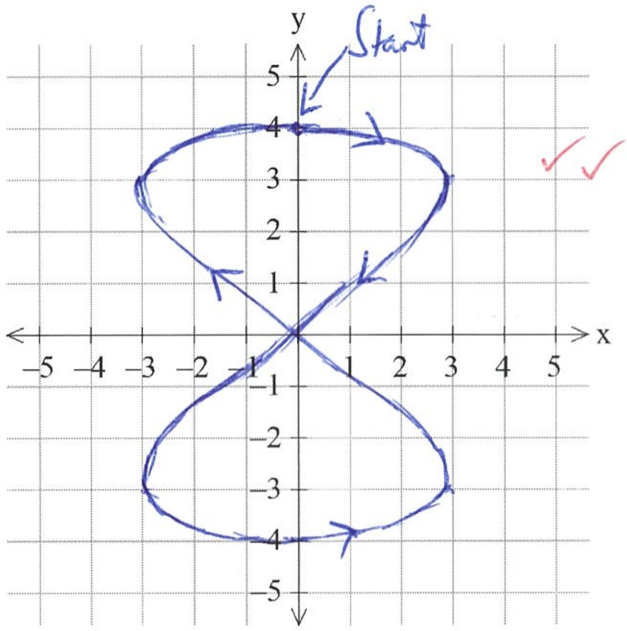
(a) Write an expression for her position $\overrightarrow{r(t)}$ given that she started at $\overrightarrow{r(0)} = 4j$

$\vec{r}(t) = 3\sin 2t\, i + 4\cos t\, j + c$

$\vec{r}(0) = 4j \Rightarrow c = 0$

i.e. $\vec{r}(t) = 3\sin 2t + 4\cos t\, j$

(b) Sketch $\overrightarrow{r(t)}$, clearly indicating the direction of travel



(c) How long does she take to complete one circuit?

Period $= \frac{2\pi}{1} = 2\pi$ units

Identify point(s) on the circuit when the acceleration is:

(d) 0 Origin $(0, 0)$ (At $t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$)

(e) directly toward the origin

at $(0, 4)$ and $(0, -4)$ (At $t = 0, \pi, 2\pi, \ldots$)